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Iteration of Positive Approximation Operators*

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We present an analysis of the limit behavior of the $k(n)$-th iterates of positive linear approximation operators L_n, as n and $k(n)$ tend to infinity. For various classes of operators the limit semigroup is explicitly identified. Two applications of the results are given: (a) identification of functions f satisfying $L_n f \geqslant f$ for all n, for a variety of operators L_n; (b) identification of the operators commuting with operators of the Bernstein type.

1. INTRODUCTION

Kelisky and Rivlin [7] studied the behavior of the iterates $B_n^{k(n)}(f; x)$ as $n \to \infty$ and $k(n) \to \infty$, where $B_n(f; x)$ denotes the n-th Bernstein polynomial mean of $f \in C[0, 1]$, i.e.,

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}$$

and $B_n{}^k(f; x)$ denotes the k-th iterate of this transform. By ad hoc computational methods these authors establish the limit formula,

$$B_n^{k(n)}(t^r; x) \to \sum_{i=1}^{r} b_i x^i, \tag{1.1}$$

(r—an arbitrarily fixed positive integer) provided $k(n)/n \to \alpha$ $(0 < \alpha < \infty)$ where

$$b_i = \frac{i}{r}\binom{r}{i}^2 \sum_{j=i}^{r} \frac{(-1)^{i+j}\binom{r-i}{j-i}^2}{\binom{2j-2}{j-i}\binom{j+r-1}{r-j}} e^{-j\alpha/2}. \tag{1.2}$$

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In the circumstance $k(n)/n \to 0$, $\| B_n^{k(n)}(f; x) - f(x)\| \to 0$ in the maximum norm topology and if $k(n)/n \to \infty$ then $\| B_n^{k(n)}(f; x) - Pf(x)\| \to 0$ where Pf denotes the projection operator $Pf(x) = f(0) + x[f(1) - f(0)]$.

With a view to place these results in a more natural context we propose to investigate the following structure. Let L_n denote a sequence of linear, positivity preserving, approximation operators acting on a linear space of functions $\mathscr{S}$. Specifically, we postulate that L_n and $\mathscr{S}$ satisfy the properties:

(a) If $f \in \mathscr{S}$ then $L_n f \in \mathscr{S}$;
(b) If $f \in \mathscr{S}, f \geqslant 0$ then $L_n f \geqslant 0$;
(c) $f(x) \equiv 1 \in \mathscr{S}$ and $L_n 1 = 1$;
(d) For each $f \in \mathscr{S}$, $L_n f - f \to 0$ where the limit relation generally signifies uniform convergence with respect to a class $\mathscr{F}$ of positive linear functionals or seminorms defined on $\mathscr{S}$, i.e., $\sup_{F \in \mathscr{F}} | F(L_n f - f)| \to 0$ as $n \to \infty$, for each $f \in \mathscr{S}$.

Our objective will be to ascertain the limit behavior of the $k(n)$-th iterates of L_n, denoted by $L_n^{k(n)}$, as n and $k(n) \to \infty$. Although several simple general propositions are available (see Section 2), in order to obtain more extensive results it becomes necessary to specify L_n concretely. We highlight two important classes of examples that will be considered in some detail later.

I. Let $\mathscr{S} = C_{2\pi}$ (the set of all continuous functions on $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$), and let

$$K_n(x) = \sum_{r=-\infty}^{\infty} c_{r,n} e^{irx}, \qquad -\pi \leqslant x \leqslant \pi$$

where $c_{r,n} \to 1$ as $n \to \infty$, for each fixed r. Furthermore, we assume $K_n(x) \geqslant 0$ and $\hat{K}_n(0) = 1/2\pi \int_{-\pi}^{\pi} K_n(x)\, dx = 1$. Put

$$(L_n f)(x) = (f * K_n)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\eta)\, K_n(x - \eta)\, d\eta, \qquad f \in C_{2\pi} \quad (1.3)$$

II. Let $\mathscr{S} = C(a, b) \cap \mathscr{G}$ where $C(a, b)$ denotes the set of continuous functions on the real interval (a, b) and $\mathscr{G}$ symbolizes some growth and continuity constraints on $f \in C(a, b)$ in the neighborhood of a and b. Let $\{X_t\}$ be a collection of random variables with the index t traversing $T = (a, b)$. Let $X_t^1, \ldots, X_t^n$ be independent, identically distributed copies of X_t. Suppose the expectation $E(X_t)$ of X_t exists and $E(X_t) = t$. Define

$$L_n(f; t) = E\left[f\left(\frac{X_t^1 + \cdots + X_t^n}{n}\right)\right], \qquad (1.4)$$

postulating the existence of these operators for each $f \in \mathcal{S}$ and $t \in T$. Observe that for $f_0(x) \equiv 1$ and $f_1(x) \equiv x$, $L_n(f_i\,; t) \equiv f_i(t)$, and that L_n is manifestly positivity preserving. Moreover, with the aid of the weak law of large numbers, provided $f \in \mathcal{S}$ obeys appropriate growth conditions, we infer that $L_n(f; t) \to f(t)$ for each $t \in T$.

Classic cases embrace the following:

(i) $T = [0, 1]$; the random variable X_t is two valued, taking the values 1 and 0 with probabilities t and $1 - t$, respectively. In this case

$$L_n(f; t) = B_n(f; t) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} t^k(1 - t)^{n-k}.$$

(ii) $T = (0, \infty)$; X_t is Poisson distributed with parameter t, i.e.,

$$P(X_t = r) = \frac{e^{-t}t^r}{r!}, \qquad r = 0, 1, \ldots .$$

Then we have

$$L_n(f; t) = M_n(f; t) = e^{-nt} \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nt)^k}{k!}, \tag{1.5}$$

the well-known Szasz–Mirakyan means.

(iii) $T = (0, \infty)$; $X_t = tX$ where X is a positive random variable satisfying $E[X] = 1$. Then

$$L_n(f; t) = U_n(f; t) = \int_0^\infty f\left(\frac{tx}{n}\right) \varphi^{(n)}(x)\, dx, \tag{1.6}$$

where $\varphi(x)$ is the density of X and $\varphi^{(n)} = \varphi * \varphi^{(n-1)}$ is the n-fold convolution of φ. Examples of the type (1.6) were discussed by Karlin [5, Chapter 6].

(iv) $T = (-\infty, \infty)$; $X_t = t + X$ where X is a real random variable satisfying $E[X] = 0$. In this case

$$L_n(f; t) = W_n(f; t) = \int_{-\infty}^{\infty} f(t + x)\, \varphi^{(n)}(x)\, dx. \tag{1.7}$$

For X normally distributed with mean zero and variance σ^2, (1.7) reduces to the Weierstrass transform.

Other probabilistic interpretations of approximation operators can be found in Renyi [12], Arato and Renyi [1], and Hille and Phillips [3].

The operator $L_n^{k_n}$ with index n and degree of iteration k_n is a familiar object of investigation in the theory of convergence of stochastic processes. The extent of iteration k_n can be regarded as a scaling of the time parameter for the n-th process. The operators L_n (and likewise $L_n^{k_n}$) have norm 1 since L_n are positive and keep fixed the unit function. The characterization and existence of the limit operator $\lim_{n\to\infty} L_n^{k_n}$ are intimately related to the theorem of Trotter (see, e.g., Yosida [14]) and Kurtz [10] for other applications) pertaining to the convergence of contraction semigroups of transformations. The Trotter theorem for discrete semigroups can be formulated as follows:

Let $\{L_n\}$ be a sequence of contraction operators on a Banach space $\mathscr{B}$ and let

$$A_n = k_n(L_n - I)$$

be the corresponding infinitesimal operator associated with L_n, where k_n is an increasing sequence of positive numbers tending to ∞. Then the following statements are equivalent:

(a) For $A = \lim_{n\to\infty} A_n$, the domain $\mathscr{D}(A)$ is dense in $\mathscr{B}$, and range $\mathscr{R}(\lambda_0 I - A)$ is also dense in $\mathscr{B}$ for some $\lambda_0 > 0$.

(b) There exists a strongly continuous contraction semigroup $T_t(t > 0)$ on $\mathscr{B}$ such that

$$\lim_{n\to\infty} \| L_n^{[tk_n]} f - T_t f \| = 0$$

for each $f \in \mathscr{B}$ and $t \in [0, \infty)$.

With the Trotter theorem as a guide, the convergence behavior of $L_n^{k_n}$ for the examples at hand is easy to determine. Indeed, consider first the sequence of convolution operators (1.3). Note that all L_n possess the common eigenfunctions e^{irx}, $r = 0, \pm 1, \pm 2, \ldots$. Exploiting this fact, we obtain that $\lim_{n\to\infty} L_n^{k_n} f$ exists if and only if for each r, $\lim_{n\to\infty} k_n(1 - c_{r,n}) = \psi(r)$ exists. (The limit behavior of $k_n(1 - c_{r,n})$ in essence determines the limit behavior of $k_n(L_n - I)$). This limit operator is embedded in a semigroup $\{V_t\}$ where for each $t > 0$ and $f \in C_{2\pi}$,

$$\lim_{n\to\infty} L_n^{[k_n t]} f(x) = V_t f(x) = (f * V_t)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, v_t(x - y)\, dy \quad (1.8)$$

and

$$v_t(x) \sim \sum_{r=-\infty}^{\infty} e^{-t\psi(r)} e^{irx}.$$

The rigorous validation of this result is easy; see Section 3.

The analysis of $\lim_{n\to\infty} L_n^{k_n}$ for the cases (i)–(iv) of II leads to other semigroups, mostly of diffusion type. Thus in case (i), if $k_n/n \to t$, $(0 < t < \infty)$ and $f \in C[0, 1]$, $f(0) = f(1) = 0$, then

$$B_n^{k_n}(f; x) \to T_t f(x) = \int_0^1 p(t; x, y) f(y)\, dy, \tag{1.9}$$

where $p(t; x, y)$ is the transition probability density of a diffusion process on $[0, 1]$ with absorbing barriers whose backward equation is

$$\frac{\partial p}{\partial t} = \frac{1}{2} x(1 - x) \frac{\partial^2 p}{\partial x^2}.$$

An explicit formula for $T_t f$ is also available, viz.,

$$p(t; x, y) = \frac{1}{y(1 - y)} \sum_{n=2}^{\infty} e^{-\frac{n(n-1)t}{2}} Q_n(x)\, Q_n(y)(n - 1)\, n(2n - 1) \tag{1.10}$$

where $Q_n(x) = x(1 - x)\, P_{n-2}^{(1,1)}(1 - 2x)$ and $P_{n-2}^{(1,1)}(x)$ is the Jacobi polynomial of parameters (1, 1), normalized to be 1 at $x = 1$. The Kelisky–Rivlin formula (1.2) can now be interpreted in terms of these classical orthogonal polynomials which comprise the natural coordinate system of the limit process. The infinitesimal generator of T_t, acting on twice continuously differentiable functions f, is the differential operator,

$$Af(x) = \tfrac{1}{2}x(1 - x) f''(x).$$

If $k_n/n \to \infty$, then for each $f \in C[0, 1]$ we have

$$B_n^{k_n}(f; x) \to T_\infty f(x) = f(0) + x[f(1) - f(0)];$$

if $k_n/n \to 0$, then $B_n^{k_n} f$ converges to $T_0 f$—the identity mapping.

The results for the other operators, (ii)–(iv), stated formally, become:

(ii) If $k_n/n \to t$ $(0 < t < \infty)$ then $M_n^{k_n} \to T_t$ where T_t is the semigroup with infinitesimal generator

$$Af(x) = \tfrac{1}{2}xf''(x), \qquad 0 < x < \infty.$$

(iii) If $k_n/n \to t$ then $U_n^{k_n} \to T_t$, whose infinitesimal generator is

$$Af(x) = \frac{\sigma^2 x^2}{2} f''(x), \qquad 0 < x < \infty,$$

where

$$\sigma^2 = \int_0^\infty (x - 1)^2\, \varphi(x)\, dx, \qquad 0 < x < \infty.$$

(iv) If $k_n/n \to t$ then $W_n^{k_n} \to T_t$, whose infinitesimal generator is

$$Af(x) = \frac{\gamma}{2} f''(x), \qquad -\infty < x < \infty,$$

where γ is an appropriate positive constant.

The ranges of validity of these limit relations are discussed in Sections 3–6. In Section 7 we further apply the technique of iteration of a family of operators in order to characterize the class $\mathscr{C}$ of functions f satisfying, for all n,

$$L_n f \geqslant f. \tag{1.11}$$

Concerning the background of this problem see Ziegler [15]. In all the cases (i)–(iv), linear functions are invariant and (1.11) prevails for convex f. Iterating (1.11) and making use of the fact that the operators are positivity preserving, lead to

$$T_t f - f \geqslant 0.$$

Dividing by t and then letting t decrease to zero we obtain

$$Af \geqslant 0, \tag{1.12}$$

provided f is in the domain of A. For cases (i)–(iv) this implies that $f''(x) \geqslant 0$ for $x \in (a, b)$. Ordinarily we assume that (1.11) holds and $f \in \mathscr{S}$, but we do not assume $f \in$ Domain A. In this circumstance, an appropriate perturbation procedure, preserving (1.11), leads to the conclusion that there exist $f_n \in$ Domain A, such that $Af_n \geqslant 0$ and $f = \lim_{n\to\infty} f_n$.

A generalized form of the characterization problem for (1.11) in the case of the Bernstein polynomial means is available. Suppose $f \in C[0, 1]$ satisfies, for all n,

$$B_n(f; x) \geqslant f(x), \qquad \text{for} \quad x = 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1.$$

Then f is convex. Similar extensions are available for some of the other approximation operators.

As a final application of the iteration technique, we identify the set of operators which commute with certain approximation operators.

For additional applications of the iteration method in determining saturation classes of approximation operators we refer to Micchelli [11].

2. Iteration of General Positive Linear Operators

Following the lines of the familiar Bohman–Korovkin arguments (cf. [9]) we indicate simple necessary and sufficient conditions for the iterates of a sequence of operators to converge to the identity operator and to a certain projection operator.

THEOREM 1. *Let* $\{L_n\}_1^\infty$ *be a sequence of positive linear operators on* $C[a, b]$ $(-\infty < a < b < \infty)$ *satisfying*

$$\begin{aligned} L_n(1; x) &\equiv 1, \qquad \text{for all } n, \\ L_n(t; x) &\equiv x, \qquad \text{for all } n. \end{aligned} \tag{2.1}$$

(a) *A necessary and sufficient condition for*

$$\lim_{n\to\infty} \| L_n^{k_n} f - Pf \|_\infty = 0 \tag{2.2}$$

where P *is the projection operator* $Pf(x) = f(a) + (x-a)/(b-a)[f(b) - f(a)]$, *is that*

$$\lim_{n\to\infty} \| L_n^{k_n} t^2 - (a+b)\,x - ab \|_\infty = 0. \tag{2.3}$$

(b) *A necessary and sufficient condition for*

$$\lim_{n\to\infty} \| L_n^{k_n} f - f \|_\infty = 0 \tag{2.4}$$

is that

$$\lim_{n\to\infty} \| L_n^{k_n} t^2 - x^2 \|_\infty = 0. \tag{2.5}$$

Remark 1. The theorem is stated with respect to norm convergence. We could just as well formulate the assertions in the following more general setting. Let $\mathscr{F}$ denote a class of positivity preserving linear functionals or seminorms. Then $\lim_{n\to\infty} \sup_{F\in\mathscr{F}} | F(L_n^{k_n} f - Pf)| = 0$ for each $f \in C[a, b]$ iff the same limit relation holds for $f(t) = t^2$.

Remark 2. Multivariate versions of the above theorem are also available. As an illustration, consider $\mathscr{S} = C(\Delta) =$ the class of continuous functions on $\Delta : \{(x, y\}, \; x \geqslant 0, \; y \geqslant 0, \; x + y \leqslant 1\}$. Let L_n denote the Bernstein polynomial means in two variables. Then the natural analog of the assertions of Theorem 1 holds.

Proof. For convenience of exposition take $a = 0$, $b = 1$.

(a) The necessity is obvious. Suppose now that (2.3) is satisfied. In view of relation (2.1) we may assume $f(0) = f(1) = 0$. It is required then to establish $\| L_n^{k_n} f \| \to 0$. Since $\| L_n^{k_n} \| = 1$ it suffices to demonstrate the convergence for a dense set in $C[0, 1]$.

Let $f \in C^1[0, 1]$ (the class of continuously differentiable functions on $[0, 1]$) satisfy $f(0) = f(1) = 0$. Clearly, there exists a constant c (depending on f but not on x such that

$$-cx(1-x) \leqslant f(x) \leqslant cx(1-x), \qquad x \in [0, 1].$$

Applying the positive operators $L_n^{k_n}$, $n = 1, 2, \ldots$, gives

$$-cL_n^{k_n} x(1-x) \leqslant L_n^{k_n} f \leqslant cL_n^{k_n} x(1-x).$$

Relation (2.1) and the hypothesis (2.3) imply that both extreme terms tend uniformly to zero and therefore $L_n^{k_n} f$ converges uniformly to zero, as well.

(b) The necessity is trivial. The sufficiency follows by a direct application to the operators $L_n^{k_n}$ of the Bohman–Korovkin theorem (see [9]).

The trigonometric analogue of Theorem 1 is weaker and amounts to a restatement of the Korovkin theorem.

THEOREM 2. *Let $\{T_n\}_1^\infty$ be a sequence of positive linear operators defined on the space $C_{2\pi}$ satisfying*

$$T_n(1; x) \equiv 1, \qquad \text{for all } n.$$

A necessary and sufficient condition for

$$\lim_{n\to\infty} \| T_n^{k_n} f - f \| = 0, \qquad \text{for all} \quad f \in C_{2\pi},$$

is that

$$\begin{aligned} \| T_n^{k_n}(\cos t) - \cos t \| \to 0, &\qquad \text{as} \quad n \to \infty, \\ \| T_n^{k_n}(\sin t) - \sin t \| \to 0, &\qquad \text{as} \quad n \to \infty. \end{aligned} \tag{2.6}$$

For more specific results which are somewhat analogous to (2.2) in the context of convolution transformations we refer to Section 3.

We close this section with a general theorem furnishing a lower bound for the extent of iteration required in order to achieve a limit different from the identity operator.

THEOREM 3. *Let $\{L_n\}_{n=1}^\infty$ be a sequence of linear operators of norm 1 such that for all f in their domain*

$$\| L_n f - f \| = O\left(\frac{1}{n^r}\right) \qquad \text{as} \quad n \to \infty.$$

If $\lim_{n\to\infty} k_n/n^r = 0$, *then*

$$\lim_{n\to\infty} \| L_n^{k_n} f - f \| = 0.$$

Proof. Clear from the inequalities

$$\| L_n^{k_n} f - f \| = \left\| \sum_1^{k_n} (L_n^j f - L_n^{j-1} f) \right\| \leqslant \sum_1^{k_n} \| L_n f - f \| = O\left(\frac{k_n}{n^r}\right) = o(1).$$

3. Trigonometric Operators of the Convolution Type

Consider a sequence of operators $\{T_n, n = 1, 2, ...\}$ acting on functions of $C_{2\pi}$, of the explicit form,

$$T_n(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) K_n(x - t) \, dt, \tag{3.1}$$

where $K_n(t)$ is a function of $C_{2\pi}$ admitting the expansion

$$K_n(t) \sim \sum_{r=-\infty}^{\infty} c_{r,n} e^{irt}, \qquad c_{0,n} = 1; \quad c_{-r,n} = \overline{c_{r,n}}. \tag{3.2}$$

Introducing the notation

$$f(x) \sim \sum_{r=-\infty}^{\infty} \hat{f}(r) e^{irx},$$

we have

$$T_n(f; x) = \sum_{r=-\infty}^{\infty} c_{r,n} \hat{f}(r) e^{irx}.$$

Throughout this section, we assume that the operators are positivity preserving, or equivalently, that $K_n(t) \geqslant 0$. This implies $\| T_n \| = 1$. For most of the subsequent results it suffices to stipulate that $\| T_n \| \leqslant C$ without necessarily T_n being positive.

Note that the numbers $\{c_{r,n}\}_{r=-\infty}^{\infty}$ comprise a full set of eigenvalues of the operator T_n with the common system (independent of n) of eigenfunctions $\{e^{irt}\}_{-\infty}^{\infty}$. Observe especially that the constant functions are invariant under all transformations T_n.

Theorem 4. *Let* $\{T_n\}$ *be a sequence of positive linear operators of the form* (3.1).

(a) *A necessary and sufficient condition for the iterates* $T_n^{k_n} f$ *to converge uniformly for every* $f \in C_{2\pi}$, *is that*

$$\lim_{n\to\infty} (c_{r,n})^{k_n} = c_r \qquad \textit{exist for all } r. \tag{3.3}$$

(b) *A necessary and sufficient condition for*

$$\lim_{n\to\infty} \| T_n^{k_n} f - \hat{f}(0) \| = 0$$

is that

$$\lim_{n\to\infty} (c_{r,n})^{k_n} = 0, \qquad \textit{for all} \quad r \neq 0. \tag{3.4}$$

(c) *A necessary and sufficient condition for*

$$\lim_{n\to\infty} \| T_n^{k_n} f - f \| = 0$$

is that

$$\lim_{n\to\infty} (c_{1,n})^{k_n} = 1. \tag{3.5}$$

(Note that (3.5) involves only the first Fourier coefficient.)

Proof. On account of the principle of uniform boundedness (a) and (b) follow by virtue of the identity

$$T_n^{k_n}(e^{irt}; x) = c_{r,n}^{k_n} e^{irx}. \tag{3.6}$$

Part (c) is Korovkin's "trigonometric" theorem [9].

The cases (b) and (c) correspond to the extremal cases where the iterates converge to the identity operator and to the projection operator $Pf = \hat{f}(0)$, on the manifold of constant functions, respectively. We next describe how to determine k_n leading to limit operators which are not projections.

Assuming that T_n, $n = 1, 2,\ldots$, are approximation operators of the form (3.1), we may write $c_{r,n} = 1 - \epsilon_{r,n}$ where $\lim_{n\to\infty} \epsilon_{r,n} = 0$ for all fixed r. We deduce immediately that the limit in (3.3) exists iff for all r, $\lim_{n\to\infty} k_n \epsilon_{r,n} = \lim_{n\to\infty} k_n(1 - c_{r,n})$ exists (with a finite imaginary part). We denote this limit by $\psi(r)$.

We can regard, for each n, the iterates $\{T_n^{[k_n t]}\} = \{\mathscr{T}_n(t)\}$, $0 < t < \infty$, as inducing a discrete time semigroup of transformations. Our objective is to determine, for k_n appropriately increasing, the limit $\lim_{n\to\infty} \mathscr{T}_n(t) = T_t$.

The preceding considerations readily yield the following:

THEOREM 5. *Let $\{T_n\}_1^\infty$ be a sequence of positive, linear approximation operators of the form* (3.1). *A necessary and sufficient condition in order that $T_n^{k_n}f$ converge uniformly to a limit for every $f \in C_{2\pi}$, is that*

$$\lim_{n\to\infty} k_n(1 - c_{r,n}) = \psi(r) \qquad \textit{exist (with } |\operatorname{Im}\psi(r)| < \infty) \textit{ for all } r. \tag{3.7}$$

If Re $\psi(r) \neq +\infty$, *for* $r \geqslant 1$, *and* $\psi(1) \neq 0$, *the limit operator is identifiable as a semigroup operator Φ_ξ such that for each $\xi > 0$ and $f \in C_{2\pi}$,*

$$\lim_{n\to\infty} \| T_n^{[k_n\xi]} - \Phi_\xi f \| = 0,$$

where $\Phi_\xi f(x)$ admits the Fourier representation

$$\Phi_\xi f(x) = \sum_{-\infty}^{\infty} e^{-\xi\psi(r)} \hat{f}(r)\, e^{irx}, \qquad f \sim \sum_{-\infty}^{\infty} \hat{f}(r)\, e^{irx}. \tag{3.8}$$

Proof. The infinitesimal generator A of Φ_ξ corresponds to the multiplier sequence operator having the structure,

$$Af(x) \sim \sum_{r=-\infty}^{\infty} (-\psi(r) \hat{f}(r))\, e^{irx},$$

provided f belongs to the domain of A.

The limit relation is readily verified for the system $\{e^{irx}\}_{r=-\infty}^{\infty}$. Therefore by the principle of uniform boundedness it is valid for $C_{2\pi}$.

When additional specifications concerning the operators are made, more exact properties of the iterates are deducible. The next two addenda indicate results of this type.

Remark 3. Let $\{T_n\}_{n=1}^\infty$ be a sequence of positive linear approximation operators of type (3.1). Assume that the coefficients $c_{r,n}$ are real and can be represented in the form,

$$c_{r,n} = 1 + \sum_{i=j(r)}^{\infty} \frac{d_{r,i}}{P_{r,i}(n)},$$

where $P_{r,i}(n)$ is a polynomial of degree i in n with leading coefficient 1 and where $j(r)$ is such that $d_{r,j(r)} \neq 0$ while $d_{r,i} = 0$ for $1 \leqslant i < j(r)$. (This form for $c_{r,n}$ covers most classical examples.) In this case the growth of k_n required for the existence of a limit semigroup is determined by $j(1)$ alone. Specifically, the limit $\lim_{n\to\infty} T_n^{k_n}f$ exists iff $\lim_{n\to\infty} k_n/n^{j(1)}$ exists.

Indeed, we check first that $d_{r,j(r)} < 0$ for all r. This follows from the fact that $\| T_n \| = 1$ for all n. Using Korovkin's theorem we can now prove that

	Fejér	de la Vallée-Poussin	Jackson	Korovkin
(A) Kernel	$\dfrac{\sin^2 \frac{nt}{2}}{n \sin^2 \frac{t}{2}}$	$1 + \dfrac{2\sum_{r=1}^{n} \binom{2n}{n+r} \cos rt}{\binom{2n}{n}}$	$\dfrac{3}{n(2n^2+1)} \left[\dfrac{\sin \frac{nt}{2}}{\sin \frac{t}{2}} \right]^4$	$\dfrac{\left\lvert \sum_{r=1}^{n+1} \sin \frac{r\pi}{n+2} e^{i(r-1)t} \right\rvert^2}{\sum_{r=1}^{n+1} \sin^2 \frac{r\pi}{n+2}}$
(B) $\lim_{n\to\infty} T_n^{(k_n)} f$ exists iff	$\lim_{n\to\infty} \frac{k_n}{n}$ exists	$\lim \frac{k_n}{n}$ exists	$\lim \frac{k_n}{n^2}$ exists	$\lim \frac{k_n}{n^2}$ exists
The function $\psi(r)$ of formulas (3.7) and (3.8) is	$\lvert r \rvert$	r^2	$\dfrac{3r^2-1}{2}$	$\dfrac{\pi^2 r^2}{2}$
If the limit in (B) is equal to ξ, the expansion of $\Phi_\xi f$ has the form	$\hat{f}(0) + 2\sum_{r=1}^{\infty} \hat{f}(r)\, e^{-\xi \lvert r \rvert} \cos rx$	$\hat{f}(0) + 2\sum_{r=1}^{\infty} \hat{f}(r)\, e^{-\xi r^2} \cos rx$	$\hat{f}(0) + 2\sum_{r=1}^{\infty} \hat{f}(r)\, e^{-\xi[\frac{3r^2-1}{2}]} \cos rx$	$\hat{f}(0) + 2\sum_{r=1}^{\infty} \hat{f}(r)\, e^{-\xi \frac{\pi^2 r^2}{2}} \cos rx$

$j(r) \geqslant j(1)$ for all r. Hence, $\lim_{n\to\infty} k_n/n^{j(1)}$ exists iff $\lim_{n\to\infty} k_n/n^{j(r)}$ exists for all r. The assertion now follows on the basis of Theorem 4.

Remark 4. For the class of cyclic variation diminishing transformations (for definitions and properties of such transformations consult Karlin [5, Chapter 9]), the following analogue of part *a*) of Theorem 1 holds:

Let $\{T_n\}_1^\infty$ be a sequence of positive linear approximation operators of the form (3.1), where $K_n(t)$ is cyclic variation diminishing of order 2. Then $\lim_{n\to\infty} \| T_n^{k_n} f - \hat{f}(0)\| = 0$, for all $f \in C_{2\pi}$, iff $\lim_{n\to\infty} k_n(1 - c_{1,n}) = \infty$.

Indeed, since for a cyclic variation diminishing transformation we have $| c_{1,n} | > | c_{r,n} |$, for all $r \geqslant 2$, the assertion is an immediate consequence of (3.6) and Theorem 4.

EXAMPLES. We tabulate below applications of the preceeding theorem for some of the classical trigonometric approximation methods. Specifically, we record the limit semigroups and indicate the appropriate iteration degree k_n associated with these approximation methods.

4. Iteration of Bernstein Polynomial Means and Related Operators

In this section we investigate the nature of the iterates of the Bernstein polynomial transforms. Also, generalizations to a class of approximation operators stemming from direct product branching processes will be briefly discussed (see [4] and [6] for a detailed treatment of direct product branching processes).

We have $B_n(1; x) \equiv 1$, $B_n(t; x) \equiv x$. Furthermore, it may be quickly verified (see, e.g., [13], p. 155) that

$$B_n(t^r; x) = \sum_{k=1}^{r} \frac{(n)_k}{n^r} c_{r,k} x^k, \qquad c_{r,r} = 1, \qquad c_{r,r-1} = \binom{r}{2}, \tag{4.1}$$

where $c_{r,k}$ are constants (independent of n) and $(n)_k = n(n-1) \cdots (n-k+1)$. Thus, the polynomials of degree $\leqslant p$ that vanish at 0 constitute an invariant manifold $\mathscr{M}_p$ for each of the transformations B_n. Letting $\sum_{i=1}^{p} a_i x^i$ be represented by a vector $a = (a_1, \ldots, a_p)$ and denoting the induced matrix transformation by $A_p(n)$ we see that the polynomial $B_n^{k_n}(t^r; x)$, $r \leqslant p$, corresponds to the vector $[A_p(n)]^{k_n} e_r^{(p)}$, where $e_r^{(p)}$ is the r-th unit vector in the p-dimensional vector space. Hence, the existence and nature of the limit are deducible from an examination of the powers of the matrices $A_p(n)$.

In view of the relation,

$$B_n^{k_n}(t^2; x) = \left(1 - \frac{1}{n}\right)^{k_n} x^2 + \left[1 - \left(1 - \frac{1}{n}\right)^{k_n}\right] x,$$

Theorem 1 yields the following conclusions:

$$\lim_{n\to\infty} \| B_n^{k_n} f - f \| = 0 \quad \text{iff} \quad \lim_{n\to\infty} k_n/n = 0,$$
$$\lim_{n\to\infty} \| B_n^{k_n} f - Pf \| = 0 \quad \text{iff} \quad \lim_{n\to\infty} k_n/n = \infty. \tag{4.2}$$

We next consider the question of existence and identification of the limits of $B_n^{k_n}$ in the general case. Note, on the basis of (4.1), that the elements $a_{rk}^{(p)}(n)$ of $A_p(n)$ have the form,

$$a_{rk}^{(p)}(n) = \begin{cases} c_{r,k} \dfrac{(n)_k}{n^r}, & 1 \leqslant k \leqslant r-2, \\ \dbinom{r}{2} \dfrac{(n)_{r-1}}{n^r}, & k = r-1, \\ \dfrac{(n)_r}{n^r}, & k = r, \\ 0, & k > r. \end{cases}$$

A simple computation reveals that

$$\lim_{n\to\infty} n(A_p(n) - I) = B_p \tag{4.3}$$

exists and the elements $(b_{rk}^{(p)})_{r,k=1}^{p}$ of B_p are given by

$$b_{rk}^{(p)} = \begin{cases} \dfrac{r(r-1)}{2}, & k = r-1, \\ -\dfrac{r(r-1)}{2}, & k = r, \\ 0, & \text{otherwise}, \end{cases}$$

so that they are independent of p. It follows that

$$\lim_{n\to\infty} n[B_n(t^r; x) - x^r] = \mathscr{B} x^r = \tfrac{1}{2} r(r-1)\, x(1-x), \qquad r = 1, 2, \dots . \tag{4.4}$$

Consider now the $p \times p$ matrices

$$[A_p(n)]^{k_n} = \left\{ \left[I + \frac{n(A_p(n) - I)}{n} \right]^n \right\}^{k_n/n}.$$

Since the expression within the curly braces, by virtue of (4.3), converges to e^{B_p}, it follows that $\lim_{n\to\infty} [A_p(n)]^{k_n}$ exists iff $\lim_{n\to\infty} k_n/n$ exists.

To sum up, the semigroups $B_n^{[k_n \xi]}$ will tend to a limit semigroup T_ξ iff we rescale time so that $k_n \sim n$. In order to identify the limit semigroup operator T_ξ, we examine its effect on the monomials x^r, $r = 1, 2, \ldots$.

We manifestly have $\lim_{n\to\infty}[A_p(n)]^{[nt]} = e^{B_p t}$. Introducing the notation

$$U_\nu(\xi; x) = \lim_{n\to\infty} B_n^{[n\xi]}(t^\nu; x) = \sum_{j=1}^{p} (e^{\xi B_p})_{\nu j}\, x^j, \qquad 1 \leqslant \nu \leqslant p,$$

we find, using (4.4), that

$$\frac{\partial}{\partial t} U_\nu(t; x) = \frac{1}{2} x(1 - x) \frac{\partial^2}{\partial x^2} U_\nu(t; x), \tag{4.5}$$

and $U_\nu(0; x) = x^\nu$.

By the principle of uniform boundedness we infer that $\lim_{n\to\infty} B_n^{[n\xi]} f(x) = T_\xi f(x)$ exists for all $f \in C[0, 1]$. Furthermore, we may write

$$T_\xi f(x) = f(0) + [f(1) - f(0)]x + \int_0^1 p(\xi; x, y) \tilde{f}(y)\, dy,$$

where $\tilde{f}(y) = f(y) - f(0) - [f(1) - f(0)]\, y$ and $p(\xi; x, y)$ is the Green's function corresponding to

$$\begin{gathered} \frac{\partial}{\partial t} p(t; x) = \frac{1}{2} x(1 - x) \frac{\partial^2}{\partial x^2} p(t; x), \\ p(t; 0) = p(t; 1) = 0. \end{gathered} \tag{4.6}$$

The explicit form of the kernel $p(t; x, y)$ is known (see [6]), viz.,

$$p(t; x, y) = \frac{1}{y(1 - y)} \sum_{n=2}^{\infty} e^{-\frac{n(n-1)t}{2}} Q_n(x)\, Q_n(y)(n - 1)\, n(2n - 1), \qquad 0 \leqslant x, \quad y \leqslant 1, \tag{4.7}$$

where $Q_n(x) = x(x - 1)\, P_{n-2}^{(1,1)}(1 - 2x)$ and $P_{n-2}^{(1,1)}(x)$ is the Jacobi polynomial of parameters (1, 1), normalized to be 1 at $x = 1$. We deduce from (4.7) that $Q_n(x)$, $n \geqslant 2$, are common eigenfunctions of the operators T_t, with associated eigenvalues $e^{-n(n-1)t/2}$, $n \geqslant 2$. We note also that 1 and x are eigenfunctions, associated with the common eigenvalue 1. The results of Kelisky–Rivlin cited in Section 1 can now be interpreted quite naturally.

We next introduce a class of approximation operators generalizing the Bernstein polynomial-means.

Let $\sum(n)$, $n \geqslant 1$, denote the set of points $\{i/n\}_{i=0}^{n}$. Consider the operator R_n mapping the functions of $C[0, 1]$, restricted to the set $\sum(n)$, into functions defined on $\sum(n)$ as follows:

$$R_n\left(f; \frac{i}{n}\right) = \sum_{j=0}^{n} P_{ij}(n) f\left(\frac{j}{n}\right),$$

where the $P_{ij}(n)$ are the transition probabilities of a Markov chain induced by a direct product branching process (see [4]). Specifically, let the absolutely monotone function $f(s) = \sum_0^\infty a_n s^n$, $a_n \geqslant 0$, $\sum_0^\infty a_n = 1$, be prescribed, and define

$$P_{ij}(n) = \frac{\text{coefficient of } z^j w^{n-j} \text{ in} f^i(z)\,[f(w)]^{n-i}}{\text{coefficient of } t^n \text{ in } f^n(t)}. \tag{4.8}$$

Some illustrations are worth recording at this point:

(a) $f(z) = e^{\lambda(z-1)}$, $\lambda > 0$. Then

$$P_{ij}(n) = \binom{n}{j}\left(\frac{i}{n}\right)^j \left(1 - \frac{i}{n}\right)^{n-j}, \qquad i, j = 0, 1,..., n.$$

(b) $f(z) = (q + pz)^n$, $0 < q = 1 - p < 1$. Then

$$P_{ij}(n) = \frac{\binom{2i}{j}\binom{2n-2i}{n-j}}{\binom{2n}{n}}.$$

(c) $f(z) = q^\alpha/(1 - pz)^\alpha$, $q = 1 - p$, $\alpha > 0$. Then

$$P_{ij}(n) = \frac{\binom{i\alpha + j - 1}{j}\binom{(n-i)\alpha + n - j - 1}{n-j}}{\binom{n\alpha + n - 2}{n}}.$$

For such operators, it is shown in [4] that the values of $R_n(Q; i/n)$, where Q is a polynomial of degree r ($r \leqslant n$), are the values taken by a polynomial of degree r at the points i/n. We can thus regard this (unique) polynomial (obtained, e.g., by Lagrange interpolation) as the transform of Q, so that R_n transforms polynomials of degree r into polynomials of degree r. For example, the Bernstein polynomial means are induced from the $P_{ij}(n)$ of example a).

Observe now that

$$\begin{gathered} R_n(1; x) \equiv 1, \qquad R_n(t; x) \equiv x \\ R_n[t(1-t); x] = \lambda_2(n)\, x(1-x), \end{gathered} \tag{4.9}$$

where

$$\lambda_2(n) = \frac{\text{coefficient of } z^{n-2} \text{ in} f^{n-2}(z)[f'(z)]^2}{\text{coefficient of } z^n \text{ in} f^n(z)}.$$

Assuming $f''(1) < \infty$ it is proved in Karlin and McGregor [6] that

$$n[1 - \lambda_2(n)] \to \gamma \tag{4.10}$$

where γ is a constant (which reduces to $f''(1)$ if f has the normalization $f'(1) = 1$).

Relations (4.9) and (4.10) imply by Theorem 1 that

$$\begin{aligned} \lim_{n\to\infty} \| R_n^{k_n} f - f \| = 0 \quad &\text{iff} \quad \lim_{n\to\infty} k_n/n = 0, \\ \lim_{n\to\infty} \| R_n^{k_n} f - Pf \| = 0 \quad &\text{iff} \quad \lim_{n\to\infty} k_n/n = \infty. \end{aligned} \tag{4.11}$$

It is established also in [6], provided $f^{(\mathrm{iv})}(1) < \infty$, that

$$\lim_{n\to\infty} n \sum_{j=0}^{n} P_{ij}(n) \left[\frac{j}{n} - \frac{i}{n}\right]^4 = 0,$$

where the convergence is uniform with respect to i, $0 \leqslant i/n \leqslant 1$. This relation is then used to prove that

$$\lim_{n\to\infty} R_n^{[nt]}(f; x) = f(0) + [f(1) - f(0)]x + \int_0^1 \tilde{p}(t; x, y) \tilde{f}(y)\, dy, \tag{4.12}$$

where $\tilde{f}(y) = f(y) - f(0) - [f(1) - f(0)]\, y$ and $\tilde{p}(t : x, y)$ is the kernel corresponding to the backward diffusion equation

$$\frac{\partial}{\partial t} p(t; x) = \frac{\gamma}{2} x(1 - x) \frac{\partial^2}{\partial x^2} p(t; x),$$
$$p(t; 0) = p(t; 1) = 0.$$

The explicit expression for $\tilde{p}(t; x, y)$ differs from that of (4.7) only in the exponent, where $1/2$ is replaced by $\gamma/2$.

5. Iterations of Certain Generalized Convolution Operators

Consider the family of approximation operators defined in II of Section 1. We shall indicate the scaling necessary so that the iterates $L_n^{k_n}$ converge in each of the examples (ii)–(iv). The formal proof of the existence of the limits can be carried out most easily with the help of suitable probabilistic interpretations. Consider first example (ii). Let N be the renewal variable for an exponential random variable with parameter 1. That is

$$N(t) = \min\{\nu \mid X_1 + \cdots + X_\nu \leqslant t < X_1 + \cdots + X_{\nu+1}\},$$

where X_i are independent, identically distributed random variables with distribution function $1 - e^{-x}$. The Szasz–Mirakyan operators can be represented as

$$L_n(f; t) = M_n(f; t) = E\left(f\left(\frac{N(nt)}{n}\right)\right),$$

where E denotes the expected value operator. Then the iterates $L_n^{k_n}$ take the form,

$$L_n^{k_n}(f; t) = E\left(\frac{f(N_1(N_2(\cdots N_{k_n}(nt)))\cdots)}{n}\right), \tag{5.1}$$

where $N_1, N_2, \ldots, N_{k_n}$ are independent copies of $N(\cdot)$. Using the explicit formula for the Laplace transform of N, the limit of (5.1) as $k_n/n \to \xi$ can be evaluated directly for the functions $\varphi_\lambda(x) = e^{-\lambda x}$ yielding

$$\lim_{n\to\infty} L_n^{[n\xi]} \varphi_\lambda(x) = \exp\left[-\frac{\lambda x}{\left(1 + \frac{\lambda\xi}{2}\right)}\right]. \tag{5.2}$$

With the aid of (5.2), the limit semigroup of $L_n^{[n\xi]}$ is now readily identified with the diffusion process whose infinitesimal generator is $Af(x) = xf''(x)/2$.

An alternative procedure is via the Trotter theorem. In applying it, we have to produce a sequence $k_n \to \infty$ such that

$$\Omega_n = k_n(L_n - I)$$

tends to a limit operator Ω and it is necessary to determine the full domain of Ω. We shall not enter into these details of the analysis.

In order to identify the limit of Ω_n we restrict attention to the subset of $\mathscr{S}$ consisting of all functions with a continuous, bounded third derivative.

We obtain

$$n[M_n(f; t) - f(t)] = ne^{-nt} \sum_{r=0}^{\infty} \left[f\left(\frac{r}{n}\right) - f(t)\right] \frac{(nt)^r}{r!}$$

$$= ne^{-nt} \sum_{r=0}^{\infty} \left[f\left(\frac{r}{n}\right) - f(t) - \left(\frac{r}{n} - t\right) f'(t)\right] \frac{(nt)^r}{r!}$$

$$= ne^{-nt} \left\{ \sum_{|r/n-t|\leqslant\delta} \left(\frac{r}{n} - t\right)^2 f''(t) \frac{(nt)^r}{r!} + \sum_{|r/n-t|\leqslant\delta} \left(\frac{r}{n} - t\right)^3 f'''(\theta_r) \frac{(nt)^r}{r!}\right.$$

$$\left. + \sum_{|r/n-t|>\delta} \left[f\left(\frac{r}{n}\right) - f(t) - \left(\frac{r}{n} - t\right) f'(t)\right] \frac{(nt)^r}{r!}\right\},$$

where $|\theta_r - t| \leqslant |r/n - t|$. Using the estimate

$$e^{-nt} \sum_{r=0}^{\infty} \left(\frac{r}{n} - t\right)^4 \frac{(nt)^r}{r!} \leqslant M_1 \frac{t^2}{n^2}$$

and standard Tchebycheff-type inequality arguments, we deduce that

$$\lim_{n\to\infty} n[M_n(f; t) - f(t)] = t\,\frac{f''(t)}{2}, \tag{5.3}$$

the convergence holding boundedly on $[0, \infty)$ and uniformly on compact subintervals of $[0, \infty)$.

The relation (5.3) persists provided f is of class $C^2[0, \infty)$ exhibiting at most polynomial growth at ∞.

Consider next the sequence of operators U_n defined in (1.6), i.e.,

$$U_n(f; x) = \int_0^\infty f\left(\frac{xy}{n}\right) \varphi^{(n)}(y)\, dy, \tag{1.6}$$

where $\varphi(y)$ is a density function of a positive random variable and $\varphi^{(n)}$ is the n-fold convolution of φ. We assume that

$$\int_0^\infty y\varphi(y)\, dy = 1 \qquad \text{and} \qquad \int_0^\infty y^4\varphi(y)\, dy < \infty.$$

An analysis paralleling the one carried out above shows that

$$\Omega_n = n(U_n - I)$$

converges to Ω where

$$\Omega f(t) = \tfrac{1}{2}\sigma^2 t^2 f''(t), \qquad \sigma^2 = \int_0^\infty (t-1)^2\, \varphi(t)\, dt \tag{5.4}$$

provided $f \in C^3[0, \infty)$ and $f''(x) = O(1/x^2)$, $f'''(x) = O(1/x^4)$ at infinity.

A probabilistic demonstration of the existence of $\lim_{n\to\infty} U_n^{[n\xi]}(f, x)$ can be carried out appealing to an appropriate version of the central limit theorem. In this manner, we obtain

$$U_n^{[n\xi]}(f; x) \to T_\xi f(x)$$

and

$$T_\xi f(x) = E_x[f(\exp(W_{\xi\sigma^2}) \cdot \exp(-\xi\sigma^2/2))]$$

where W_t, $t \geqslant 0$, denotes the standard Brownian motion process and E_x symbolizes the expected value of the indicated random variable where initially $W_0 = x$. When stringent moment conditions are imposed on φ then the limit relation $U_n^{[n\xi]}f \to T_\xi f$ prevails also for functions of exponential growth and minimal type. We discuss some aspects of these results in the next section.

For the operators of example (iv) (see (1.7)) we find that the limit of $L_n^{[n\xi]}$ coincides with the standard Brownian motion with infinitesimal generator $\Omega = (\sigma^2/2)\, d^2/dx^2$ where σ^2 is the variance of the convoluted random variable.

A variant of the last example of some interest arises in the following way. Let X be a random variable with density function $\varphi(x)$, and assume its first four moments exist. Define the operators

$$\Phi_n(f; t) = E\left[f\left(t - \frac{X}{c_n}\right)\right] = c_n \int_{-\infty}^{\infty} f(x)\, \varphi(c_n(t - x))\, dx, \qquad (5.5)$$

where $c_n > 0$ are such that $\lim_{n\to\infty} c_n = \infty$.

Assuming boundedness of the first three derivatives of the function f, we find that

$$\begin{aligned}\lim_{n\to\infty} c_n{}^2[\Phi_n(f; t) - f(t)] &= \lim_{n\to\infty} \frac{f''(t)}{2}\, c_n{}^2[\Phi_n(x^2; t) - t^2] \\ &= \frac{\sigma^2}{2} f''(t),\end{aligned}$$

where σ^2 is the variance of X. Trotter's theorem suggests that the correct rescaling is $T_n^{[c_n{}^2 t]}$ and the infinitesimal generator is $\Omega f = (\sigma^2/2) f''(t)$. The rigorous validation of these assertions is quite straightforward.

6. Approximations for Certain Operators of Exponential Growth

Let $U_n(f; t)$ denote the convolution operators defined in (1.6). Imposing an additional requirement on $\varphi(x)$ permits the extension of the validity of the convergences $U_n(f; t) \to f(t)$ and $U_n^{[n\xi]}(f; t) \to T_\xi f$ to a larger class of functions.

The condition on $\varphi(x)$ is expressed in terms of the moment generating function,

$$V(s) = \int_0^\infty e^{s\xi} \varphi(\xi)\, d\xi. \qquad (6.1)$$

Specifically, assume that

$$c = \sup\{s \mid V(s) < \infty\} > 0$$

so that φ essentially decays exponentially fast at $+\infty$. *Let Γ designate the class of continuous function on* $[0, \infty)$ *obeying a growth constraint at* ∞ *of the form,*

$$|f(t)| < Ke^{(c-\alpha)t}, \qquad 0 \leqslant t < \infty, \tag{6.2}$$

for some positive constants K and α.

THEOREM 6. *Let the operators $U_n(f; t)$ be defined as in* (1.6). *Then for each $f \in \Gamma$,*

$$\lim_{n\to\infty} U_n(f; t) = f(t), \qquad 0 \leqslant t < \infty, \tag{6.3}$$

where the convergence is uniform on compact subsets.

Proof. Consider the interval $[0, M]$ where M is an arbitrary positive number, and let t be an arbitrary fixed point inside this interval. Given $\epsilon > 0$, determine $\delta = \delta(\epsilon, M)$, $\delta < M$, such that if $|x - y| \leqslant \delta$ and $0 \leqslant x, y \leqslant M$ then $|f(x) - f(y)| < \epsilon/4$.

We have

$$|U_n(f; t) - f(t)| \leqslant \int_{t|x/n-1|>\delta} \left|f\left(\frac{tx}{n}\right) - f(t)\right| \varphi^{(n)}(x)\, dx + \frac{\epsilon}{4}.$$

Let $A = A(M, f)$ be such that $|f(t)| \leqslant A$ for $0 \leqslant t \leqslant M$. Since $\int_0^\infty x\varphi(x)\, dx = 1$, it follows for all $n > N_1 = N_1(\epsilon, M)$, by virtue of the weak law of large numbers, that

$$|U_n(f; t) - f(t)| \leqslant \frac{\epsilon}{2} + \int_{M(x/n-1)>\delta} \left|f\left(\frac{tx}{n}\right)\right| \varphi^{(n)}(x)\, dx. \tag{6.4}$$

The estimate required for the last term is formulated as a separate lemma, since it will be needed in the sequel.

LEMMA. *Let $\varphi(t)$ and Γ be defined as previously. Let f be a function in Γ and let $M > 0$ and $b > 1$ be arbitrarily prescribed. Then there exists a constant $B = B(M, b)$ and an $N = N(M)$ such that for $n > N$,*

$$\int_{bn}^\infty \left|f\left(\frac{tx}{n}\right)\right| \varphi^{(n)}(x)\, dx \leqslant B[m(b)]^n, \qquad 0 \leqslant t \leqslant M, \tag{6.5}$$

where

$$m(u) = \inf_s e^{-us} V(s), \qquad -\infty < u < \infty, \tag{6.6}$$

and $m(u) < 1$ for $u > 1$.

Proof. Using the estimate (6.2) and performing an obvious change of variables, we have

$$\int_{bn}^{\infty} \left| f\left(\frac{tx}{n}\right) \right| \varphi^{(n)}(x)\, dx \leqslant K \int_{b}^{\infty} e^{(c-\alpha)tx} n\varphi^{(n)}(nx)\, dx.$$

Integration by parts produces

$$K\left[-e^{(c-\alpha)tu} \int_{u}^{\infty} n\varphi^{(n)}(nx)\, dx\right]_{b}^{\infty} + K(c-\alpha)t \int_{b}^{\infty} e^{(c-\alpha)xt}[1 - \phi^{(n)}(nx)]\, dx \tag{6.7}$$

where $\phi(x) = \int_0^x \varphi(\xi)\, d\xi$.

The first term of (6.7) can be put in probabilistic language, viz.,

$$e^{(c-\alpha)tu} \int_{u}^{\infty} d\phi^{(n)}(nx) = e^{(c-\alpha)tu} Pr\{S_n \geqslant nu\} \tag{6.8}$$

where $S_n = \sum_{i=1}^{n} X_i$ and the X_i's are independent, identically distributed random variables with distribution function $\phi(x)$.

The bounds developed in [2] confirm the inequality

$$Pr\{S_n \geqslant nu\} \leqslant [m(u)]^n, \tag{6.9}$$

for all $u > 1$.

Since $e^{-us}V(s)$ is convex in s, we may infer for $u > 1$, that $m(u)$ is endowed with the following properties:

(i) $m(u) < 1$,
(ii) $m(u) = e^{-ut_u}V(t_u)$, where $t_u > 0$,
(iii) $m(v) < m(u)$, if $v > u > 1$, (6.10)
(iv) for each $A > 1$ there exists a $d = d(A)$, $0 < d < c$ (where c is the constant defined in (6.1)), such that $m(u) \leqslant e^{-du}m(d)$ for $u \geqslant A$.

Relations (6.9) and (6.10) imply that the left side of (6.8) does not exceed

$$[m(d)]^n\, e^{u[(c-\alpha)t-dn]},$$

provided $u \geqslant A > 1$. Therefore, an $N_1 = N_1(M)$ exists such that for $n > N_1$ the quantity (6.8) vanishes when $u \to \infty$. Referring to relation (6.7), we find for $0 \leqslant t \leqslant M$ and $n > N_1(M)$ that the left side of (6.5) is bounded by

$$K\left[e^{M_1 b} Pr\{S_n \geqslant nb\} + M_1 \int_{b}^{\infty} e^{M_1 x}[1 - \phi^{(n)}(nx)]\, dx\right], \tag{6.11}$$

where $M_1 = M(c - \alpha)$. Now

$$\int_b^\infty e^{M_1 x}[1 - \phi^{(n)}(nx)]\,dx = \sum_{k=1}^{\infty} \int_{kb}^{(k+1)b} e^{M_1 x}(1 - \phi^{(n)}(nx)]\,dx$$

$$\leqslant \sum_{k=1}^{\infty} e^{M_1(k+1)b} b[1 - \phi^{(n)}(nkb)]$$

$$= be^{M_1 b} \sum_{k=1}^{\infty} e^{M_1 kb} Pr\left\{\frac{S_n}{n} \geqslant kb\right\}$$

$$\leqslant be^{M_1 b} \sum_{k=1}^{\infty} e^{M_1 kb}[m(kb)]^n,$$

where the last inequality results by appeal to (6.9).

Putting together these estimates yields the following upper bound for the left side of (6.5):

$$K_1 \sum_{k=1}^{\infty} e^{M_1 kb}[m(kb)]^n = K_1[m(b)]^n \sum_{k=1}^{\infty} e^{M_1 kb} \left[\frac{m(kb)}{m(b)}\right]^n. \tag{6.12}$$

Now let $t_b > 0$ be defined by $m(b) = e^{-bt_b}V(t_b)$. Then

$$m(kb) \leqslant e^{-kbt_b}V(t_b)$$

and (6.12) is bounded from above by

$$K_1[m(b)]^n \sum_{k=1}^{\infty} e^{M_1 kb} \left[\frac{e^{-kbt_b}}{e^{-bt_b}}\right]^n = K_1[m(b)]^n e^{bM_1} \sum_{k=0}^{\infty} e^{(M_1 - nt_b)kb}$$

$$= B(M, b)[m(b)]^n$$

since the last sum is finite for n sufficiently large. The proof of the lemma is now complete.

Returning to the proof of the theorem, we deduce from (6.4) and the Lemma, that there exists an $N_0 = N_0(\epsilon, M)$ and a constant $C = C(\epsilon, M)$ such that for $n > N_0$,

$$|\,U_n(f; t) - f(t)| \leqslant \frac{\epsilon}{2} + C(\epsilon, M)\left[m\left(1 + \frac{\delta}{M}\right)\right]^n.$$

Observing that $m(1 + \delta/M) < 1$, we infer the existence of $N = N(\epsilon, M)$ such that for $n > N(\epsilon, M)$,

$$|\,U_n(f; t) - f(t)| < \epsilon, \qquad \text{for all} \quad 0 \leqslant t \leqslant M,$$

and the proof of Theorem 6 is complete.

Assertions similar to that of Theorem 6 hold for the other examples of approximation operators introduced in Section 1. We underscore this phenomenon by stating the relevant results in the case of the Szasz-Mirakyan operators.

Let $\mathscr{S}$ designate the class of continuous functions f on $[0, \infty)$ for which there exist positive constants $A = A_f$ and $B = B_f$ satisfying

$$|f(t)| \leqslant Ae^{Bt}, \qquad t \geqslant 0.$$

Then for each $f \in \mathscr{S}$,

$$\lim_{n\to\infty} M_n(f; t) = f(t), \qquad 0 \leqslant t < \infty,$$

where the convergence is uniform on compact subsets.

We can establish that the iterates $L_n^{[k_n \xi]}$ converge for all functions belonging to the classes considered in this section. The proofs involve adaptations and refinements of the previous methods. We do not enter into details but illustrate the convergence by considering the much simpler situation of functions of polynomial growth.

Let Π_+ be the class of functions f of $C[0, \infty)$ exhibiting at most polynomial growth at $x = +\infty$, i.e., for each $\epsilon > 0$ there exists an r and a polynomial $P(t)$ such that

$$|f(t) - P(t)| \leqslant \epsilon(1 + t^r), \qquad 0 \leqslant t < \infty.$$

Consider first a monomial $f(x) = x^r$, $r \geqslant 2$. A simple computation, facilitated by probabilistic interpretations, shows that

$$U_n(x^r; t) = \frac{t^r}{n^r} E\left[\left(\sum_{i=1}^{n} X_i\right)^r\right]$$

$$= t^r \left[1 + \frac{1}{n}\binom{r}{2}(\mu_2 - 1) + O\left(\frac{1}{n^2}\right)\right],$$

where $\mu_2 = \int_0^\infty t^2\varphi(t)\,dt > [\int_0^\infty t\varphi(t)\,dt]^2 = 1$.

It follows that

$$U_n^{[n\xi]}(x^r; t) = t^r \left[1 + \frac{c}{n} + O\left(\frac{1}{n^2}\right)\right]^{[n\xi]}, \qquad 0 < \xi < \infty,$$

where $c = (\mu_2 - 1)\binom{r}{2} > 0$. The convergence is manifestly established for the case of monomials. Since $U_n^{k_n}$ are positive transformations of norm 1 the proof of convergence of $U_n^{[n\xi]}f$ for all f in Π_+ is accomplished in a straightforward manner. The extension of this property to $f \in \Gamma$ requires further technical devices.

7. THE ITERATION METHOD AND CHARACTERIZATIONS OF CONVEXITY

In this section we investigate the problem of characterizing the functions f satisfying

$$L_n f \geqslant f, \qquad \text{for all} \quad n, \tag{7.1}$$

where L_n is a sequence of approximation operators of the types considered previously. In most of the cases we shall find that the inequalities (7.1) imply that f is convex. The converse is immediate provided L_n maps linear functions into themselves and is positivity preserving. Some results concerning this problem were uncovered earlier by Ziegler [15].

A central role in Ziegler's article is played by the notion of "strongly centered distributions" which designates roughly an exponential decay of the convolutions of the underlying distributions away from the mean. We shall apply the iteration technique in order to yield further characterizations of convex functions under wider conditions. The key ideas underlying the analysis of this section were described in the introductory section.

We commence by proving a characterization theorem for functions satisfying (7.1) which are twice continuously differentiable.

THEOREM 7. *Let L_n, $n = 1, 2,\ldots,$ be a sequence of positive linear operators defined on $C[a, b]$, satisfying*:

(i) $L_n(1; x) \equiv 1$, $L_n(t; x) \equiv x$, *for all* n.

(ii) $\lim_{n\to\infty} L_n(t^2; x) \equiv x^2$.

(iii) *For each x_0 such that for n sufficiently large $L_n[(t - x_0)^2; x_0] > 0$, we have*

$$\lim_{n\to\infty} \frac{L_n[(t - x_0)^4; x_0]}{L_n[(t - x_0)^2; x_0]} = 0. \tag{7.2}$$

If $f \in C^2[a, b]$ satisfies (7.1), *then f is convex.*

Remark 1. Notice that Theorem 7 can be reformulated as a local property.

Remark 2. Conditions (i)–(iii) are fulfilled for examples (i)–(iv) of Section 1 and, in fact, whenever $L_n^{[k_n\xi]}$ converges to a limit semigroup T_ξ corresponding to a diffusion stochastic process (i.e., such that Ω is essentially a local operator).

Remark 3. If L_n is "strongly centered" in the sense of Ziegler [15] then (iii) certainly holds.

Proof of Theorem 7. Suppose $f \in C^2[a, b]$ satisfies (7.1). Assume to the contrary that f is not convex. Then for some x_0, $\alpha = f''(x_0) < 0$. We infer, for an appropriate $c > 0$, that

$$f(x) - l(x) \leqslant \frac{\alpha}{2}[(x - x_0)^2 - c(x - x_0)^4]$$

for all $x \in [a, b]$, where $l(x)$ is the linear function $f(x_0) + (x - x_0)f'(x_0)$. Since $L_n \geqslant 0$ it follows that

$$L_n[f - l; x_0] \leqslant \frac{\alpha}{2}\{L_n[(t - x_0)^2; x_0] - cL_n[(t - x_0)^4; x_0]\}, \tag{7.3}$$

and the right side is negative for n sufficiently large by virtue of condition (iii) and since $\alpha < 0$. On the other hand, $L_n(f; x_0) \geqslant f(x_0)$ so that $L_n(f - l; x_0) \geqslant f(x_0) - l(x_0) = 0$. The manifest contradiction implies the desired conclusion.

Theorem 7 can be extended to include functions defined on infinite intervals provided growth restrictions are imposed on f to insure the validity of (7.3).

Remark 4. If instead of the limit relation (7.2) we assume

$$\lim_{n\to\infty} \frac{L_n[(t - x_0)^{2r}; x_0]}{L_n[(t - x_0)^2; x_0]} = 0, \tag{7.4}$$

for some integer $r \geqslant 2$, then the conclusion of Theorem 7 persists.

Remark 5. If it is desired to weaken the differentiability condition in Theorem 7 without use of "strong centeredness" then it must be required that (7.1) holds for all $x \in [a, b]$. In this circumstance a smoothing or perturbation device is employed as illustrated in the next theorem.

THEOREM 8. *Let U_n denote the approximation operators* (1.6), *where the density function $\varphi(x)$ possesses $2r$ moments* ($r \geqslant 2$). *Let $f(x)$ be defined on* $[0, \infty)$ *and grow at ∞ slower than x^{2r}. Suppose*

$$U_n(f; x) \geqslant f(x) \qquad \text{a.e., } \textit{for all} \quad n. \tag{7.5}$$

Then f is convex a.e.

Proof. Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables with density $\varphi(x)$ and $\xi_1, \xi_2, \ldots$ independent, identically and exponentially distributed random variables. Clearly,

$$U_n(f; t) = E\left[f\left(\frac{X_1 + \cdots + X_n}{n}\, t\right)\right].$$

Consider next

$$\psi_m(t) = E\left[f\left(\frac{\xi_1 + \cdots + \xi_m}{m}\, t\right)\right].$$

We have

$$U_n(\psi_m\,; t) = E\left\{f\left[\left(\frac{X_1 + \cdots + X_n}{n}\right)\left(\frac{\xi_1 + \cdots + \xi_m}{m}\right) t\right]\right\}$$

and

$$U_n(\psi_m\,; t) \geqslant \psi_m(t) \qquad \text{for all } t \tag{7.6}$$

because of (7.5).

It is easy to check that $\psi_m(t) \in C^2[0, \infty)$ and $\psi_m(t)$ has a polynomial growth slower than t^{2r}. Observe also that

$$L_n[(t - x_0)^2; x_0] = \frac{\sigma^2 x_0^{\,2}}{n}, \qquad \sigma^2 = \int_0^\infty (t-1)^2\, \varphi(t)\, dt$$

and

$$L_n[(t - x_0)^{2r}; x_0] = O\left(\frac{1}{n^r}\right),$$

so that (7.4) is manifestly satisfied. The argument of the proof of Theorem 7 shows that $\psi_m(t) = \psi_m(f; t)$ is convex. Letting $m \to \infty$ we get $\lim_{m\to\infty} \psi_m(f; t) = f(t)$ a.e. so that f is a.e. equal to a convex function. The proof of Theorem 8 is complete.

Remark 6. A variety of refinements and extensions of Theorem 8 are accessible. For example, we may merely assume that $\sigma^2 < \infty$ without further moment conditions. With only a second moment assumption the condition (7.2) is not meaningful. Nevertheless, assuming that f is bounded and satisfies (7.5) it can be established that f is convex a.e.

Remark 7. The corresponding characterizations of convexity in case L_n are the operators (1.5) or (1.7) also hold.

The following theorem depends more decisively on the iteration method.

THEOREM 9. *Let f be a function of $C[0, 1]$ such that*

$$B_n\left(f; \frac{i}{n}\right) \geqslant f\left(\frac{i}{n}\right), \qquad i = 0, 1, \ldots, n; \quad n = 0, 1, \ldots . \tag{7.7}$$

Then f is convex. (B_n denote, as previously, the Bernstein polynomial operators).

Proof. Let x be an arbitrary point of [0, 1]. Choose a sequence of rational points $i_r/n_r \to x$. We find that

$$\lim_{n_r\to\infty} B_{n_r}^{[n_r t]} f\left(\frac{i_r}{n_r}\right) = T_t f(x).$$

It follows on the basis of (7.7) that

$$T_t f(x) \geqslant f(x), \qquad 0 \leqslant x \leqslant 1.$$

Applying the positive operator T_s and using its semigroup properties we obtain

$$T_t g\,(x) \geqslant g_s(x), \qquad 0 \leqslant x \leqslant 1, \tag{7.8}$$

where $g_s(x) = 1/s \int_0^s T_t f(x)\, ds$ belongs to the domain of

$$\Omega = \frac{1}{2}\, x(1-x) \frac{d^2}{dx^2},$$

the infinitesimal generator of T_t. But (7.8) implies that $\Omega g_s(x) \geqslant 0$ and therefore $g_s(x)$ is convex. Letting $s \to 0^+$, clearly $g_s(x) \to f(x)$ so that $f(x)$ is convex as well.

Remark 9. It is clear from the proof that an assumption that $f(x)$ admits only discontinuities of the first kind would suffice.

Remark 10. The proof is valid, with obvious modifications, for the operators R_n arising from direct product branching processes.

Remark 11. The result of Theorem 9 persists for the Szasz–Mirakyan operators where (7.7) now becomes

$$M_n\left(f; \frac{i}{n}\right) \geqslant f\left(\frac{i}{n}\right), \qquad i, n = 0, 1, 2,...,$$

and f is assumed to grow at infinity slower than an exponential.

8. Operators Commuting with the Bernstein Operators

In this section we apply the iteration technique with view to determine the form of the operators that commute with all Bernstein operators. An identical result holds for operators commuting with the approximation operators induced by direct product branching processes (see Section 4). The characterization in the Bernstein polynomials case was first obtained by Konheim and Rivlin [8].

Let U be an operator mapping $C[0, 1]$ into itself and satisfying

$$UB_n = B_nU. \tag{8.1}$$

Define $Pf(x) = (1 - x)\, Uf(0) + xUf(1)$. One immediately verifies that $PB_n = B_nP$. Clearly, $V = U - P$ commutes with B_n and satisfies $Vf(0) = Vf(1) = 0$. Iteration of B_n leads to the relation

$$VT_\xi = T_\xi V, \tag{8.2}$$

where T_ξ is the semigroup of operators given in (1.9), (see also (1.10)). Analysis of formula (1.10) reveals that the polynomials $Q_n(x)$, $n \geqslant 2$, constitute the only common eigenfunctions of T_ξ, with associated eigenvalues $e^{-n(n-1)/2)\xi}$. Observe that

$$e^{-\frac{n(n-1)}{2}\xi}\, VQ_n = VT_\xi Q_n = T_\xi VQ_n\,,$$

which demonstrates that $VQ_n(x)$ is also an eigenfunction corresponding to the eigenvalue $e^{-n(n-1)\xi/2}$. It follows that

$$VQ_n(x) = c_nQ_n(x), \qquad \text{for all } n, \tag{8.3}$$

for some constants c_n. Inspection of the formula

$$Q_n(x) = x(1 - x)\, P_{n-2}^{(1,1)}(1 - 2x)$$

indicates that $Q_n(x)$ is even or odd about $x = 1/2$ according to whether n is even or odd. Moreover, classical properties of $P_{n-2}^{(1,1)}(1 - 2x)$ entail that $Q_n(\frac{1}{2}) \neq 0$ for n even and $Q_n(1/3) \neq 0$ for n odd. (Use of the last fact can be circumvented at the expense of a more expansive argument.)

In view of (8.3), we have

$$\begin{aligned} B_2VQ_n &= c_nB_2Q_n = 2c_nQ_n(\tfrac{1}{2})\, x(1 - x), && n \text{ even},\\ B_3VQ_n &= c_nB_3Q_n = 6c_nQ_n(\tfrac{1}{3})\, x(1 - x)(1 - 2x), && n \text{ odd}. \end{aligned} \tag{8.4}$$

On the other hand,

$$\begin{aligned} VB_2Q_n &= 2Q_n(\tfrac{1}{2})\, c_2x(1 - x), && n \text{ even},\\ VB_3Q_n &= 6Q_n(\tfrac{1}{3})\, c_3x(1 - x)(1 - 2x), && n \text{ odd}. \end{aligned} \tag{8.5}$$

Comparison of (8.4), (8.5), and the commutativity relation implies

$$\begin{aligned} c_n &= c_2\,, && n \text{ even},\\ c_n &= c_3\,, && n \text{ odd}. \end{aligned} \tag{8.6}$$

Now let f be any polynomial. Representing

$$\tilde{f}(x) = f(x) - (1 - x) f(0) - x f(1)$$

as a linear combination of the $Q_n(x)$ we readily deduce that

$$Vf = V\tilde{f} = c\left[\frac{\tilde{f}(x) + \tilde{f}(1 - x)}{2}\right] + d\left[\frac{\tilde{f}(x) - \tilde{f}(1 - x)}{2}\right].$$

Thus, we have

$$Vf = ax + b + \tilde{c}f(x) + \tilde{d}f(1 - x),$$

where the constants a and b are linear functionals on f and $\tilde{c}$ and $\tilde{d}$ are constants depending on the approximation operators (in this case, the Bernstein polynomial means).

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